2013 Collaborative Problem-Solving Contest
April 10th, 2013 – April 17th, 2013

This test consists of fifteen extended problems to be solved by your entire school in one week. Answers must be exact, complete, simplified, and written in the appropriate blanks on the answer sheet. Each problem on this test is worth 12 points. If a problem consists of multiple parts, the score for the problem shall be $S = \left\lfloor 12\left(\frac{c}{t}\right) \right\rfloor$, where c is the number of parts the team gets correct, t is the total number of parts, and $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

1. Young Dogs Need New Tricks

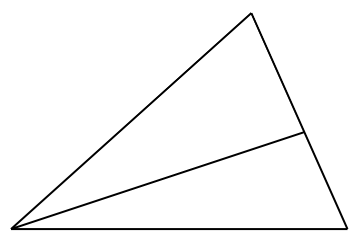
Design the "best" event for an elementary face-to-face math contest. Some aspects to consider:

Presentation: written, oral, displayed, electronic
Difficulty: for participants, administrators, graders
Timing: per problem, round, or test, maybe variable
Fun: for participants, observers, administrators, maybe graders
Response: multiple-choice/free-response/matching, oral/written/displayed/electronic
Non-Math Features: physical activity, trivia, inter-dependency, gamesmanship, pop culture

Of course there are many other aspects you could consider, and other dimensions to those listed.

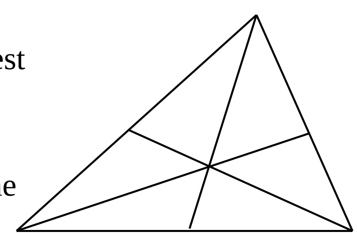
Please send clymer@natassessment.com an e-mail with the subject "CPSC #1 Submission" and with a thorough description of your event attached as a .docx file with a filename that includes your school name and city. Your description should at a minimum discuss how problems are presented, how answers are submitted, and how graders score answers. All submissions will be judged (admittedly somewhat subjectively) on completeness, creativity, cleverness, and presentation. NA&T reserves the right to be inspired by your submission.

2. Cevians



a. In the triangle shown to the right with one cevian, all segment lengths between two points of intersection are integers. What is the smallest possible sum of all segment lengths?

b. In the triangle shown to the right with one cevian, all segment lengths between two points of intersection are ***distinct*** integers. What is the smallest possible sum of all segment lengths?



c. In the triangle with three concurrent cevians shown to the right, all segment lengths around the perimeter are integers. What is the smallest possible value of the perimeter?

d. In the triangle with three concurrent cevians shown to the right, all segment lengths around the perimeter are ***distinct*** integers. What is the smallest possible value of the perimeter?

e. In the triangle with three concurrent cevians shown above and to the right, the areas of all regions are integers. What is the smallest possible value of the total area?

f. In the triangle with three concurrent cevians shown above and to the right, the areas of all regions are ***distinct*** integers. What is the smallest possible value of the total area?

3. Self-Referential Multiple-Choice Test

Each of the parts below asks a question, with four possible answers given. The complete set of answers to all parts is unique, with all answers making true statements. The "answer" to each part below is either "a", "b", "c", or "d". In the question "What is $1+1$?", with answers "a) 1 b)2 c)3 d)2", the answer is either "b" or "d" (subject to the influence of other questions), not "2".

A. What is the answer to Part G?
a) a b) d c) c d) b

B. What is the answer to Part I?
a) d b) a c) b d) c

C. Which answer occurs exactly four times?
a) c b) b c) d d) a

D. Which answer occurs the most?
a) d b) c c) c d) a

E. What is the answer to part N?
a) c b) a c) d d) b

F. What is the answer to part B?
a) a b) c c) a d) b

G. What is the nearest part prior to this one with an answer of "b"?
a) A b) B c) C d) D

H. Which answer occurs exactly eight parts apart at least once?
a) a b) d c) b d) c

I. What is the answer to part F?
a) b b) c c) b d) a

J. Which answer occurs exactly two times?
a) c b) b c) d d) a

K. What is the answer to this part?
a) b b) c c) b d) d

L. Which answer to a part is followed by a consecutive answer to a consecutive part? For example, if the answer to Part C were "b" and the answer to Part D were "c", "b" could be an answer to this problem. Consider "d" and "a" to be consecutive in that order.
a) a b) d c) c d) c

M. What is the first part with an answer of "a"?
a) B b) C c) D d) A

N. What is the answer to part E?
a) a b) d c) a d) c

O. Which part has the same answer as this part?
a) D b) C c) B d) A

4. Nomenclature

Of the numbers in this problem,

a. which is the biggest? b. which has the largest magnitude? c. which is the greatest?

$-1.6180304*10^7$	977/572	192809
7855719	42	145505
-154312	1.644905	-6553722
2.7182805	-196914	-2987698045

5. Two-Digit Numbers

a. How many positive two-digit numbers could be elements of a set of five positive two-digit integers that collectively use all ten digits?

b. How many sets of five positive two-digit integers use all ten digits?

c. What is the smallest number that is the product of five positive two-digit integers that collectively use all ten digits?

d. What is the smallest number that is divisible by five two-digit integers that collectively use all ten digits?

e. What is the smallest number that is divisible by five two-digit integers that collectively use all ten digits, no two of which are multiples of one another?

f. What is the smallest number that is divisible by five two-digit integers that collectively use all ten digits, no two of which have a two-digit common factor?

g. What is the smallest number that is divisible by five two-digit integers that collectively use all ten digits, each of which is relatively prime to at least one of the others?

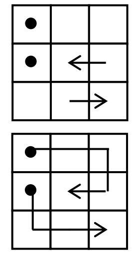
h. What is the smallest number that is divisible by five two-digit integers that collectively use all ten digits, each of which is relatively prime to all of the others?

i. What is the smallest positive integer that is divisible by nine positive two-digit integers starting with each digit 1-9?

j. What is the smallest positive integer that is divisible by nine positive two-digit integers starting with each digit 1-9, no two of which are multiples of one another?

k. What is the smallest positive integer that is divisible by nine positive two-digit integers starting with each digit 1-9, no two of which have a two-digit common factor?

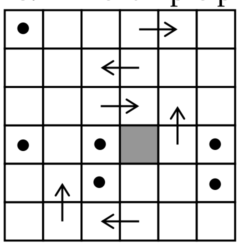
6. Directrix



Z ↓ →
Y → ↓ ←

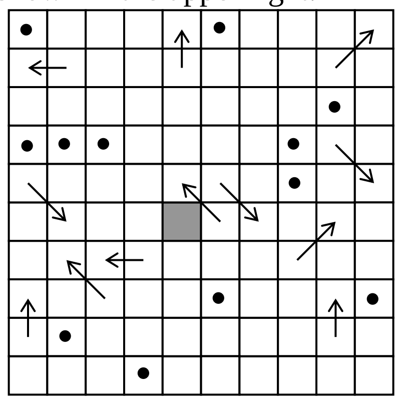
In the grids below, the starting points (dots) and endpoints (arrowheads) of a set of crooked arrows are given. The directions that the sections of each arrow point are given to the right of the grid; the arrows are not necessarily in order, but the sections of each arrow ***are*** in order from left to right within each arrow's description. All squares of the grid contain part of an arrow, no arrows cross one another, arrows only turn at the centers of squares, and arrows only go in multiples of 45 degree angles. The gray cells have no significance, other than that the answer to each part will be an alphabetical list of the lines which pass through the gray cell or any of the eight adjacent cells. An example problem and its solution is shown in the upper right.

a.



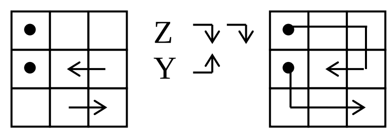
Z ↘ ↑ →
Y ↓ ← ↖ ↓ ←
X ↑↘ →
W← ↙↓→↑
V ↑↙←
U ↗ ↘↑

b.

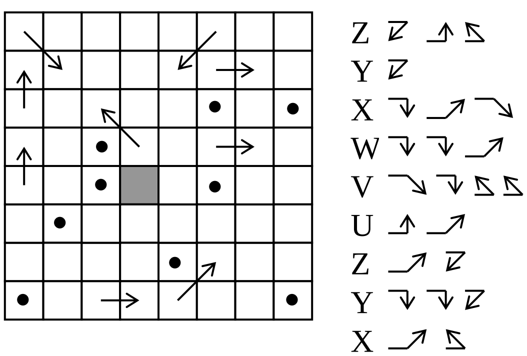


Z ↗↓ ← ↘
Y ↘ → ↖
X → ↙ ← ↗ ←
W → ↖ ↑ ↘
V ← ↘ → ↗
U ↖ ↑
T ↗ ↘ ↙ ←
S ↓ → ↑ ↘
R ↓ ← ↖
Q ← ↑
P ↘ ↓ ← ↙ → ↑
O → ↙ ← ↘ → ↗

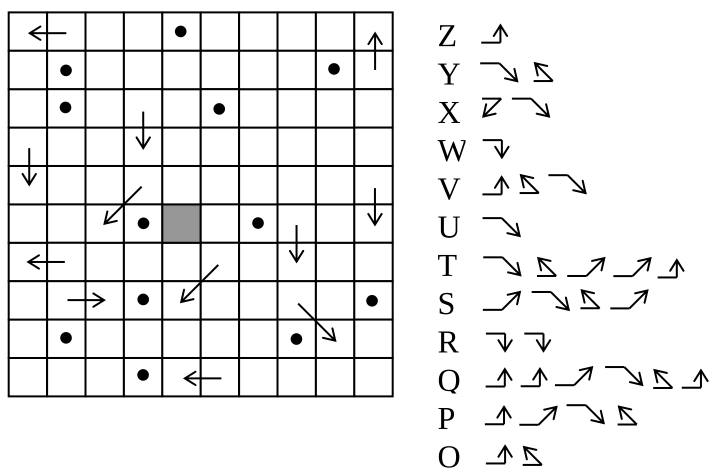
The grids below are similar to those above, but the ***angles*** of the turns are given, instead of the direction of the arrow segments. An example problem and its solution are shown to the right.



c.



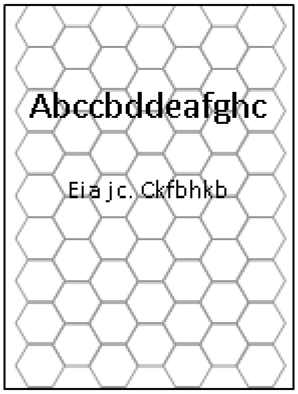
d.



7. Cyptogramese Technical Bookstore

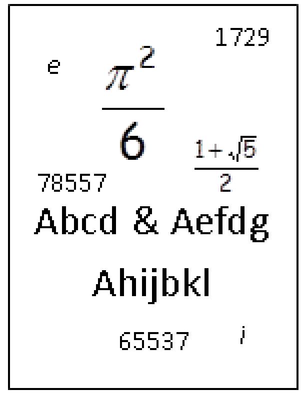
Your view through the store window is shown below. What are the titles & subtitles of the books? The words on each book's cover have been encrypted using different mappings.

a.



b.

c.



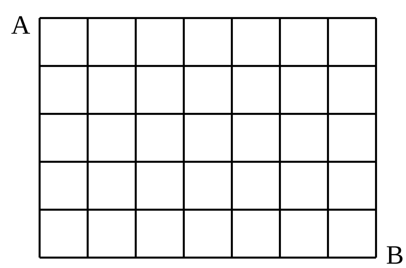
d.

e.

8. Walking on the Gridlines

a. What musician would be most likely to record a song with the same title as this problem?

Parts b through h refer to paths from A to B in the seven-by-five array of unit squares to the right. Note that a path may pass through a segment or intersection multiple times in any direction, but may only turn at the intersection of two line segments.



How many paths from A to B

b. have a length of 12? c. have a length of 15?

d. have a length of 18? e. turn exactly once?

f. turn exactly twice? g. turn exactly three times?

h. have a length of 20 and turn exactly four times?

9. Points Near Points

a. What is the maximum number of points that can be chosen in a plane such that when any three points are chosen, at least one pair of points has a separation of one unit?

b. What is the maximum number of points that can be chosen in a plane such that when any four points are chosen, at least one pair of points has a separation of one unit?

c. What is the maximum number of points that can be chosen in a plane such that when any five points are chosen, at least one pair of points has a separation of one unit?

d. What is the maximum number of points that can be chosen in a plane such that when any three points are chosen, at least two pairs of points have a separation of two units? The two pairs may share a point or not.

e. What is the maximum number of points that can be chosen in a plane such that when any four points are chosen, at least two pairs of points have a separation of two units? The two pairs may share a point or not.

f. What is the maximum number of points that can be chosen in a plane such that when any five points are chosen, at least two pairs of points have a separation of two units? The two pairs may share a point or not.

g. What is the maximum number of points that can be chosen in space such that when any three points are chosen, at least one pair of points has a separation of one unit?

h. What is the maximum number of points that can be chosen in space such that when any four points are chosen, at least one pair of points has a separation of one unit?

i. What is the maximum number of points that can be chosen in space such that when any five points are chosen, at least one pair of points has a separation of one unit?

j. What is the maximum number of points that can be chosen in space such that when any three points are chosen, at least two pairs of points have a separation of two units? The two pairs may share a point or not.

k. What is the maximum number of points that can be chosen in space such that when any four points are chosen, at least two pairs of points have a separation of two units? The two pairs may share a point or not.

l. What is the maximum number of points that can be chosen in space such that when any five points are chosen, at least two pairs of points have a separation of two units? The two pairs may share a point or not.

10. Haunted Hall of Mirrors

You’re a paranormal investigator who has learned that vampires and ghosts are hunting humans in the local Hall of Mirrors, have already infiltrated the facility, and will strike at midnight. Each “room” in a hall of mirrors contains either a vampire, a ghost, a human, or a double-sided mirror arranged diagonally across the room, and each of these occupies the same number of rooms. You quickly circle each hall and note how many “people” you see when you look in each door. Knowing that vampires to not appear in mirrors and ghosts only appear in mirrors, you must determine the products of the room numbers where the humans are, so that they can be evacuated in time.

a.

	2	1	1	1	
2	1	2	3	4	0
0	5	6	7	8	4
	0	1	3	1	

b.

	1	1	
0	1	2	1
2	3	4	2
	1	1	

c.

	1	1	0	1	
1	1	2	3	4	1
	1	1	0	1	

d.

	3	1	2	4	
2	1	2	3	4	2
1	5	6	7	8	1
2	9	10	11	12	2
	0	3	2	0	

e.

	1	2	2	3	
4	1	2	3	4	4
1	5	6	7	8	2
1	9	10	11	12	4
2	13	14	15	16	1
	0	1	5	3	

f.

		1	1	
	1\1	1	2	1
1	3	4	5	2
1	6	7	8	1
	1	2	1	

g.

		1	1	
	0\2	1	2	2
0	3	4	5	0
2	6	7	8	2
	2	1	3	

h.

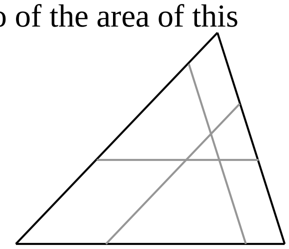
		1	1	
	1\1	1	2	1
1	3	4	5	0
2	6	7	8	2
	1	1	2	

i.

			0	0	
		0	1	2	0
	1	1\2	3	4	2
2	5	6	7	8	4
2	9	10	11	12	2
	1	1	2	3	

11. Divided Triangle

a. Three line segments are drawn in a triangle parallel to the three sides. Considered on its own, each line segment divides the triangle into two regions with equal areas. Together, the three line segments border a triangle similar to the original triangle. What is the ratio of the area of this similar triangle to that of the original triangle?



b. This part is similar (pun intended :-) to the first part, except that when each line segment is considered on its own, it divides the triangle into two regions, one of which is twice the area of the other. What is the sum of all possible values of the ratio of the area of the similar triangle to that of the original triangle?

12. Garden Paths

In the gardens below, flowers will be placed on the numbered squares and tiles will be placed on other squares subject to the following constraints:

- The tiles in each garden will form a single path that can branch and weave, but it cannot have any loops, nor can it contain any two-by-two blocks of tiles. It is okay if there are untiled regions without flowers.
- The numbers in the flowers' squares indicate how many consecutively adjacent squares are tiled. For a multi-digit number such as "24," this is actually multiple one-digit numbers representing multiple sets of consecutively adjacent squares. E.g. "24" indicates that in the 8 adjacent squares, there exist four consecutively adjacent tiles that make up one part of the path and two consecutively adjacent tiles that make up another part of the path. In this case, there will be exactly 6 tiles adjacent to these flowers.

For each part of this problem, the answer will be the maximum number of tiles that can make up the path in each garden.

a.

		11							
					24				
		13					122		
2					5				
	3							23	
4				5		33			
								31	
	3								

b.

1										12	
			33				1111				
		24					23			15	
			11		11						
	5		4								13
								1			
				4			14				
										4	
	111				2						

c.

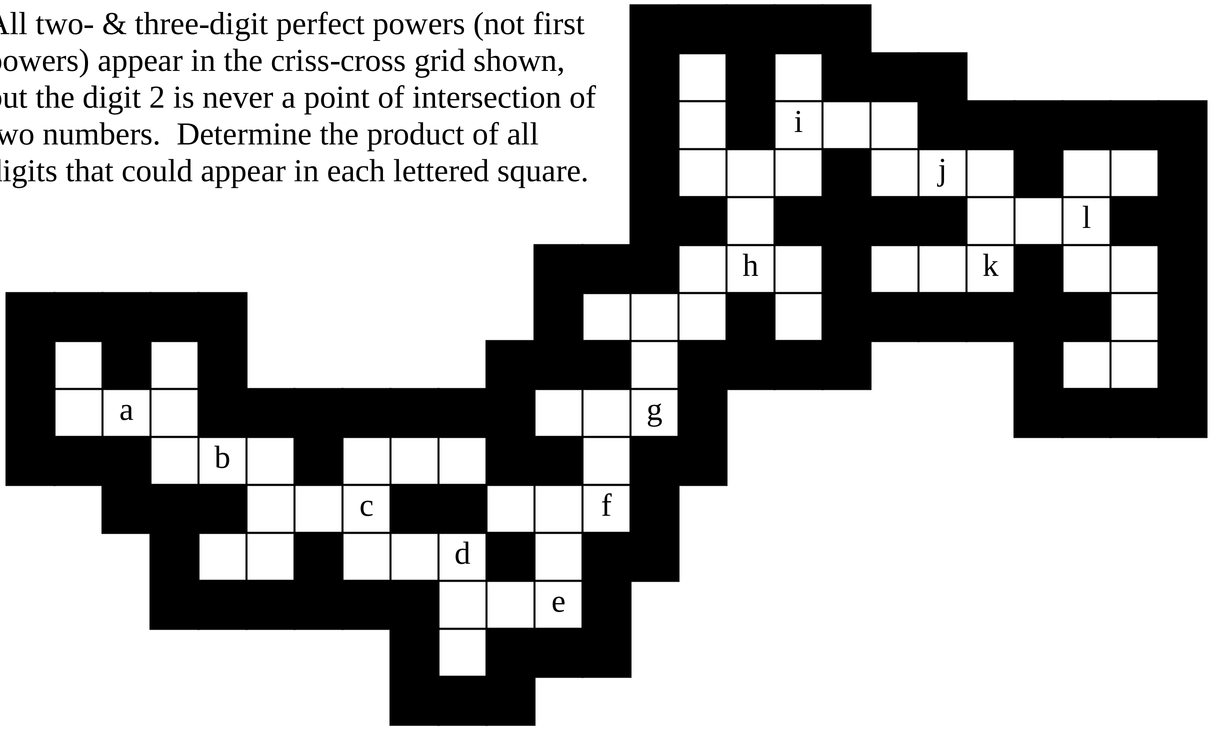
	4	12										11	3	
			13				15			33				
4				6										4
							111				5			
4					22			4			22			
			112											12
								113		113			24	
		4			5									
	23			2							4			
				111			13	4			12			4
	7			12			22							
											112			3
			22			24		24			14			
12		14											6	
				4		12				3				

d.

	1			21			
		6				3	
				1			
	3					22	

13. Perfect Power Criss-Cross

All two- & three-digit perfect powers (not first powers) appear in the criss-cross grid shown, but the digit 2 is never a point of intersection of two numbers. Determine the product of all digits that could appear in each lettered square.



14. Subsets

a. A subset of the set of integers from 1 to 100 has the property that the absolute value of the difference between an element and twice another element is always greater than 1. What is the maximum number of elements in the subset?

b. A subset of the set of integers from 1 to 100 has the property that any pair of numbers in the subset has a greatest common factor that is a prime number. What is the maximum number of elements in the subset?

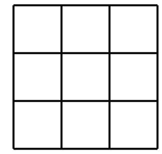
c. A subset of the set of integers from 1 to 100 has the property that the product of any pair of numbers in the subset is not a factor of 1080. What is the maximum number of elements in the subset?

d. A subset of the set of integers from 1 to 1000 has the property that the positive difference between any pair in the subset is not between 4 and 7 inclusive. What is the maximum number of elements in the subset?

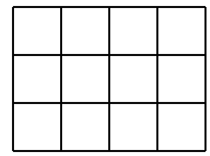
e. A subset of the set of integers from 1 to 10000 has the property that the positive difference between any pair in the subset is not a multiple of seven or eleven. What is the maximum number of elements in the subset?

f. A subset of the set of integers from 1 to 10000 has the property that the product of any pair of numbers in the subset is not a multiple of 12. What is the maximum number of elements in the subset?

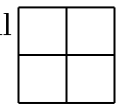
15. Special Shapes



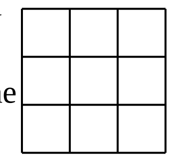
a. A three-by-three Super Square using the numbers 1-9 exactly once each has the property that each row, column, and diagonal has a sum that is a multiple of 5, but there is at least one row, one column, and two diagonals with a sum other than 15. How many Super Squares are there? Do not count rotations and/or reflections of a Super Square as a different Super Square.



b. A three-by-four Rad Rectangle using twelve distinct positive integers has the property that each row and column has the same average value, as do two of the four "diagonals" (45 degree line of three cells). How many Rad Rectangles have the minimum possible value of the largest entry? Do not count reflections of a Rad Rectangle as a different Rad Rectangle.



c. A two-by-two Splendid Square using distinct positive integers has the property that all the rows, columns, and diagonals have different sums with a collective greatest common factor greater than one. What is the smallest possible product of the numbers?



d. A three-by-three Splendid Square using distinct positive integers has the property that all the rows, columns, and diagonals have different sums with a collective greatest common factor greater than one. What is the smallest possible product of the numbers?